DIFFERENTIAL EQUATION SOLUTION STRATEGIES

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https://intuitiveexplanations.com/math/differential-equation-solution-strategies/

This document is a compilation of various strategies for solving differential equations. Proofs or derivations are presented for most of the methods discussed. The relevant definitions and properties concerning the various types of differential equations are also presented. At the end of each section, there is a subsection that explains when the different methods are usually applicable. Many of the methods presented in this document are based on the textbook I used in my Differential Equations class (*Differential Equations and Their Applications* by Martin Braun, fourth edition).

Contents

1. First-Order Differential Equations

1.1. **Characterization of First-Order Equations.** A **first-order differential equation** is an equation of the form

$$\frac{dy}{dt} = f(t, y).$$

If an additional equation of the form

$$y(t_0) = y_0$$

is also given, then the two equations together are called an **initial-value problem**. Every initial-value problem has a unique solution; if an initial condition is not provided, the general solution will have one arbitrary constant.[1] A first-order differential equation of the form

$$\frac{dy}{dt} + p(t)y = q(t) \tag{1}$$

[1] This requires that f be sufficiently well-behaved.

is called **linear**. If $q(t) = 0$ then the equation is of the form

$$\frac{dy}{dt} + p(t)y = 0 \tag{2}$$

and it is called **homogeneous**; otherwise, it is called **nonhomogeneous**.

If $y_1(t)$ is a solution of (2) that is not always equal to zero, then the general solution to (2) is

$$y(t) = Cy_1(t).^2$$

If $y_1(t)$ is a solution of (2) that is not always equal to zero and $\psi(t)$ is a solution of (1), then the general solution to (1) is

$$y(t) = Cy_1(t) + \psi(t).^3$$

1.2. **Direct Integration.** The simplest possible differential equation is of the form

$$\frac{dy}{dt} = f(t).$$

Integrating both sides with respect to t gives the general solution

$$y(t) = \int f(t)\,dt + C.$$

If we are given the initial condition $y(t_0) = y_0$, then we may find a particular solution immediately, without first finding the general solution, in the following manner:

$$\begin{aligned} \int_{t_0}^{t} y(t)\,dt &= \int_{t_0}^{t} f(t)\,dt \\ y(t) - y_0 &= \int_{t_0}^{t} f(t)\,dt \\ y(t) &= y_0 + \int_{t_0}^{t} f(t)\,dt. \end{aligned}$$

1.3. **Homogeneous Linear Equations.** To solve the general homogeneous first-order linear differential equation, we may proceed in the following manner:

$$\begin{aligned} \frac{dy}{dt} + p(t)y &= 0 \\ \frac{dy}{dt} &= -p(t)y \\ \frac{dy/dt}{y} &= -p(t) \\ \int \frac{dy/dt}{y}\,dt &= -\int p(t)\,dt + C \\ \int \frac{dy}{y} &= -\int p(t)\,dt + C \\ \ln|y| &= -\int p(t)\,dt + C \\ |y| &= e^{-\int p(t)\,dt + C} \\ |y| &= Ce^{-\int p(t)\,dt}. \end{aligned}$$

[2]This requires that p be sufficiently well-behaved.

[3]This requires that p and q be sufficiently well-behaved.

Now, y must have the same sign for all t, because otherwise the integral on the left-hand side would cross the singularity at $y = 0$. Consequently, by adjusting the value of C appropriately, we may still eliminate the absolute value bars from y. The general solution is then

$$y = Ce^{-\int p(t)\,dt}.$$

If we are given the initial condition $y(t_0) = y_0$, then we may find a particular solution immediately, without first finding the general solution, in the following manner:

$$\begin{aligned}
\frac{dy}{dt} + p(t)y &= 0 \\
\frac{dy/dt}{y} &= -p(t) \\
\int_{t_0}^{t} \frac{dy/dt}{y}\,dt &= -\int_{t_0}^{t} p(t)\,dt \\
\int_{y_0}^{y} \frac{dy}{y} &= -\int_{t_0}^{t} p(t)\,dt \\
\ln\left|\frac{y}{y_0}\right| &= -\int_{t_0}^{t} p(t)\,dt \\
\left|\frac{y}{y_0}\right| &= e^{-\int_{t_0}^{t} p(t)\,dt}.
\end{aligned}$$

Now, y_0 and y must have the same sign, because otherwise the integral on the left-hand side would cross the singularity at $y = 0$. Thus the quantity y/y_0 is positive, and since the right-hand side is also positive it follows that we may eliminate the absolute value bars. Multiplying by y_0 then gives the general solution:

$$y = y_0 e^{-\int_{t_0}^{t} p(t)\,dt}.$$

1.4. **Nonhomogeneous Linear Equations.** We will now solve the general nonhomogeneous first-order linear differential equation

$$\frac{dy}{dt} + p(t)y = q(t). \tag{3}$$

First, note that

$$\frac{d}{dt}\left[ye^{f(t)}\right] = e^{f(t)}\left[\frac{dy}{dt} + f'(t)y\right].$$

If we let $f(t) = \int p(t)\,dt$ then we obtain

$$\frac{d}{dt}\left[ye^{\int p(t)\,dt}\right] = e^{\int p(t)\,dt}\left[\frac{dy}{dt} + p(t)y\right].$$

This suggests multiplying both sides of (3) by the quantity $e^{\int p(t)\,dt}$. Doing so allows us to proceed in the following manner:

$$\begin{aligned}
e^{\int p(t)\,dt}\left[\frac{dy}{dt} + p(t)y\right] &= e^{\int p(t)\,dt}q(t) \\
\frac{d}{dt}\left[ye^{\int p(t)\,dt}\right] &= e^{\int p(t)\,dt}q(t) \\
ye^{\int p(t)\,dt} &= \int e^{\int p(t)\,dt}q(t)\,dt + C \\
y &= e^{-\int p(t)\,dt}\left[\int e^{\int p(t)\,dt}q(t)\,dt + C\right].
\end{aligned}$$

If we are given the initial condition $y(t_0) = y_0$, then we may find a particular solution immediately, without first finding the general solution, in the following manner:

$$\begin{aligned}
e^{\int p(t)\,dt}\left[\frac{dy}{dt} + p(t)y\right] &= e^{\int p(t)\,dt}q(t) \\
\frac{d}{dt}\left[ye^{\int p(t)\,dt}\right] &= e^{\int p(t)\,dt}q(t) \\
ye^{\int p(t)\,dt} - y_0\left[e^{\int p(t)\,dt}\right]_{t=t_0} &= \int_{t_0}^{t} e^{\int p(t)\,dt}q(t)\,dt \\
ye^{\left[\int p(t)\,dt\right]_{t=t}} &= y_0 e^{\left[\int p(t)\,dt\right]_{t=t_0}} + \int_{t_0}^{t} e^{\int p(t)\,dt}q(t)\,dt \\
y &= y_0 e^{\left[\int p(t)\,dt\right]_{t=t_0} - \left[\int p(t)\,dt\right]_{t=t}} + e^{\int p(t)\,dt}\int_{t_0}^{t} e^{\int p(t)\,dt}q(t)\,dt \\
y &= y_0 e^{-\int_{t_0}^{t} p(t)\,dt} + e^{-\int p(t)\,dt}\int_{t_0}^{t} e^{\int p(t)\,dt}q(t)\,dt.
\end{aligned}$$

1.5. **Separable Equations.** A first-order differential equation of the form

$$\frac{dy}{dt} = \frac{g(t)}{f(y)}$$

is called **separable**. This equation may be solved easily by moving the $f(y)$ term to the left-hand side and integrating:

$$\begin{aligned}
f(y)\frac{dy}{dt} &= g(t) \\
\int f(y)\frac{dy}{dt}\,dt &= \int g(t)\,dt \\
\int f(y)\,dy &= \int g(t)\,dt.
\end{aligned}$$

In general, it is not possible to find an explicit formula for y in terms of t.

If we are given the initial condition $y(t_0) = y_0$, then we may find a particular solution immediately, without first finding the general solution, in the following manner:

$$\begin{aligned}
f(y)\frac{dy}{dt} &= g(t) \\
\int_{t_0}^{t} f(y)\frac{dy}{dt}\,dt &= \int_{t_0}^{t} g(t)\,dt \\
\int_{y_0}^{y} f(y)\,dy &= \int_{t_0}^{t} g(t)\,dt.
\end{aligned}$$

Again, it is not possible in general to find an explicit formula for y in terms of t.

1.6. **Exact Equations.** Consider the first-order differential equation

$$f_t(t,y) + f_y(t,y)\frac{dy}{dt} = 0,$$

where f_t and f_y are functions of two variables. To solve this differential equation, observe that

$$\frac{d}{dt}f(t,y) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

according to the multivariable chain rule. Hence, if there exists a function $f(t,y)$ such that $\partial f/\partial t = f_t$ and $\partial f/\partial y = f_y$, then we may rewrite the differential equation as

$$\frac{d}{dt} f(t,y) = 0.$$

Integrating both sides then easily gives the general solution as

$$f(t,y) = C.$$

In general, it is not possible to find an explicit formula for y in terms of t.

Differential equations for which this strategy works—that is, for which such an f exists—are called **exact**.

1.6.1. *Finding a function f.* From vector analysis, we know that such a function f exists if and only if

$$\frac{\partial f_t}{\partial y} = \frac{\partial f_y}{\partial t}.^{4}$$

If this equation is satisfied, then we may find f in three different ways. The first way, which is typically the easiest, is to note that

$$\begin{aligned} f(t,y) &= \int f_t(t,y)\,dt + C(y) \\ &= \int f_y(t,y)\,dy + C(t) \end{aligned}$$

and then determine the functions $C(y)$ and $C(t)$, up to a constant, by pattern recognition.

The second way, which is useful if the integral $\int f_y(t,y)\,dy$ is difficult to compute, is to start with the first relation from the first method,

$$f(t,y) = \int f_t(t,y)\,dt + C(y),$$

and differentiate both sides with respect to y:

$$f_y(t,y) = \frac{\partial}{\partial y} \int f_t(t,y)\,dt + C'(y).$$

Subtracting the quantity $\frac{\partial}{\partial y} \int f_t(t,y)\,dt$ from both sides gives the equation

$$C'(y) = f_y(t,y) - \frac{\partial}{\partial y} \int f_t(t,y)\,dt,$$

which may be integrated to find

$$C(y) = \int f_y(t,y) - \frac{\partial}{\partial y} \int f_t(t,y)\,dt\,dy + C.$$

Thus, we find that

$$f(t,y) = \int f_t(t,y)\,dt + \int f_y(t,y) - \frac{\partial}{\partial y} \int f_t(t,y)\,dt\,dy + C.$$

Any terms of $f_y(t,y)$ containing both t and y will be canceled by the term $\frac{\partial}{\partial y} \int f_t(t,y)\,dt$, which may simplify the integration. Note also that this formula will give an incorrect answer if $\partial f_t/\partial y \neq \partial f_y/\partial t$, so check this first.

[4] This requires that f_t and f_y be sufficiently well-behaved.

On the other hand, if the integral $\int f_t(t,y)\,dt$ is difficult to compute, we may use the third method, which is analogous to the second. We start with the second relation from the first method,

$$f(t,y) = \int f_y(t,y)\,dy + C(t),$$

and differentiate both sides with respect to t:

$$f_t(t,y) = \frac{\partial}{\partial t}\int f_y(t,y)\,dy + C'(t).$$

Subtracting the quantity $\frac{\partial}{\partial t}\int f_y(t,y)\,dy$ from both sides gives the equation

$$C'(t) = f_t(t,y) - \frac{\partial}{\partial t}\int f_y(t,y)\,dy,$$

which may be integrated to find

$$C(t) = \int f_t(t,y) - \frac{\partial}{\partial t}\int f_y(t,y)\,dy\,dt.$$

Thus, we find that

$$f(t,y) = \int f_y(t,y)\,dy + \int f_t(t,y) - \frac{\partial}{\partial t}\int f_y(t,y)\,dy\,dt.$$

Any terms of $f_t(t,y)$ containing both t and y will be canceled by the term $\frac{\partial}{\partial t}\int f_y(t,y)\,dt$, which may simplify the integration. Note also that this formula will give an incorrect answer if $\partial f_t/\partial y \neq \partial f_y/\partial t$, so check this first.

1.6.2. *Making an equation exact.* The obvious shortcoming of our work in the previous section is that if there is no function $f(t,y)$ such that $\partial f/\partial t = f_t$ and $\partial f/\partial y = f_y$, then we are stuck. In particular, as we stated earlier, if

$$\frac{\partial f_t}{\partial y} \neq \frac{\partial f_y}{\partial t},$$

then the differential equation is not exact. However, consider multiplying both sides of the equation by an arbitrary function $\mu(t,y)$, to obtain

$$f_t(t,y)\mu(t,y) + f_y(t,y)\mu(t,y)\frac{dy}{dt} = 0.$$

This differential equation is exact if

$$\begin{aligned}\frac{\partial}{\partial y}\Big[f_t(t,y)\mu(t,y)\Big] &= \frac{\partial}{\partial t}\Big[f_y(t,y)\mu(t,y)\Big]\\ \frac{\partial f_t}{\partial y}\mu + f_t\frac{\partial \mu}{\partial y} &= \frac{\partial f_y}{\partial t}\mu + f_y\frac{\partial \mu}{\partial t}.\end{aligned}$$

Unfortunately, this is a partial differential equation that we cannot solve for $\mu(t,y)$ in general. However, there are two special cases: if μ is a function of t alone or a function of y alone. If we assume that μ is a function of t alone, then $\partial\mu/\partial y = 0$ and we obtain

$$\frac{\partial f_t}{\partial y}\mu = \frac{\partial f_y}{\partial t}\mu + f_y\frac{\partial \mu}{\partial t},$$

which can be solved for $\partial\mu/\partial t$ to find

$$\frac{\partial \mu}{\partial t} = \left(\frac{\partial f_t/\partial y - \partial f_y/\partial t}{f_y}\right)\mu.$$

If the right-hand side is not a function of t alone, then this equation has no solution. If it is, however, then this is a homogeneous first-order linear differential equation that can be solved to obtain

$$\mu(t) = \exp\left(\int \frac{\partial f_t/\partial y - \partial f_y/\partial t}{f_y}\,dt\right).$$

On the other hand, if we assume that μ is a function of y alone, then $\partial\mu/\partial t = 0$ and we obtain

$$\frac{\partial f_t}{\partial y}\mu + f_t\frac{\partial \mu}{\partial y} = \frac{\partial f_y}{\partial t}\mu,$$

which can be solved for $\partial\mu/\partial y$ to find

$$\frac{\partial \mu}{\partial y} = \left(\frac{\partial f_y/\partial t - \partial f_t/\partial y}{f_t}\right)\mu.$$

If the right-hand side is not a function of y alone, then this equation has no solution. If it is, however, then this is a homogeneous first-order linear differential equation that can be solved to obtain

$$\mu(y) = \exp\left(\frac{\partial f_y/\partial t - \partial f_t/\partial y}{f_t}\right).$$

Having solved for μ, we can multiply both sides of the original differential equation by it to make the equation exact. Then we may find a function f as before.

1.7. **What to Do in General.** The strategies discussed in sections 1.3–1.5 are all special cases of the strategy in section 1.6. Hence, all first-order linear differential equations that can be solved with the strategies discussed in section 1 may be solved according to section 1.6. However, to minimize work, the following strategy is advisable:

(1) If the equation is linear and homogeneous, use the strategy of section 1.3.
(2) If the equation is linear and nonhomogeneous, use the strategy of section 1.4.
(3) If the equation is separable, use the strategy of section 1.5.
(4) Otherwise, use the strategy of section 1.6.

2. Second-Order Differential Equations

2.1. **Characterization of Second-Order Equations.** A **second-order differential equation** is an equation of the form

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right).$$

If two additional equations of the form

$$y(t_0) = y_0$$
$$y'(t_0) = y'_0$$

are also given, then the three equations together are called an **initial-value problem**. Every initial-value problem has a unique solution; however, if initial conditions are not provided, the general solution will have two arbitrary constants.[5] A second-order differential equation of the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \tag{4}$$

is called **linear**. If $g(t) = 0$ then the equation is of the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0 \tag{5}$$

and it is called **homogeneous**; otherwise, it is called **nonhomogeneous**.

[5]This requires that f be sufficiently well-behaved.

Two functions $y_1(t)$ and $y_2(t)$ are called **linearly independent** if neither is a scalar multiple of the other.[6] The **Wronskian** of two functions $y_1(t)$ and $y_2(t)$ is the quantity

$$W\left[y_1, y_2\right](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

The Wronskian of two linearly independent solutions of (5) is nonzero for all t, while the Wronskian of two linearly dependent solutions of (5) is zero for all t.

If $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of (5), then the general solution to (5) is

$$y(t) = C_1 y_1(t) + C_2 y_2(t).^7$$

If $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of (5) and $\psi(t)$ is a solution of (4), then the general solution to (4) is

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + \psi(t).^8$$

2.2. **Direct Reduction of Order.** If y does not appear in a second-order differential equation, i.e. it is of the form

$$\frac{d^2y}{dt^2} = f\left(t, \frac{dy}{dt}\right),$$

then letting $v = dy/dt$ converts it to a first-order equation:

$$\frac{dv}{dt} = f(t, v).$$

One constant of integration is introduced in the solution of this first-order equation, and a second is introduced in recovering y:

$$y(t) = \int v(t)\, dt.$$

In the context of an initial-value problem, the initial condition $y'(t_0) = y_0'$ is used in the solution of the first-order equation and the initial condition $y(t_0) = y_0$ is used in recovering y.

2.3. **Reduction of Order.** Suppose that $y_1(t)$ is a nonzero solution of the homogeneous second-order linear differential equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0, \tag{6}$$

and suppose that we would like to find a second, linearly independent, solution $y_2(t)$. Since both $y_1(t)$ and $y_2(t)$ are nonzero, it follows that

$$y_2(t) = y_1(t)v(t)$$

for some function $v(t)$. The first and second derivatives of $y_2(t)$ follow from the product rule:

$$\begin{aligned} \frac{dy_2}{dt} &= y_1'(t)v(t) + y_1(t)v'(t) \\ \frac{d^2y_2}{dt^2} &= y_1''(t)v(t) + y_1'(t)v'(t) + y_1'(t)v'(t) + y_1(t)v''(t) \\ &= y_1''(t)v(t) + 2y_1'(t)v'(t) + y_1(t)v''(t). \end{aligned}$$

Plugging these into (6), we find that:

$$y_1''(t)v(t) + 2y_1'(t)v'(t) + y_1(t)v''(t) + p(t)y_1'(t)v(t) + p(t)y_1(t)v'(t) + q(t)y_1(t)v(t) = 0$$

[6] As the scalar multiple may be 0, this means that neither function may be the zero function.

[7] This requires that p and q be sufficiently well-behaved.

[8] This requires that p, q, and g be sufficiently well-behaved.

$$y_1(t)v''(t) + \Big[2y_1'(t) + p(t)y_1(t)\Big]v'(t) + \Big[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)\Big]v(t) = 0.$$

Now, since $y_1(t)$ is a solution of (6), we know that $y_1''(t) + p(t)y_1'(t) + q(t)y_1(t) = 0$, and hence

$$y_1(t)v''(t) + \Big[2y_1'(t) + p(t)y_1(t)\Big]v'(t) = 0.$$

To put the equation in standard form, we divide through by $y_1(t)$, which is nonzero:

$$v''(t) + \left[2\left(\frac{y_1'(t)}{y_1(t)}\right) + p(t)\right]v'(t) = 0.$$

Using the strategy of section 2.2, we let $w(t) = v'(t)$ and obtain:

$$w'(t) + \left[2\left(\frac{y_1'(t)}{y_1(t)}\right) + p(t)\right]w(t) = 0.$$

This is a homogeneous first-order linear differential equation, and its solution[9] via the techniques of section 1.3 is:

$$\begin{aligned}
w(t) &= \exp\left(-\int 2\left(\frac{y_1'(t)}{y_1(t)}\right) + p(t)\,dt\right) \\
&= \frac{\exp\left(-\int p(t)\,dt\right)}{\exp\left(2\int \frac{y_1'(t)}{y_1(t)}\,dt\right)} \\
&= \frac{\exp\left(-\int p(t)\,dt\right)}{\Big[\exp\big(\ln|y_1(t)|\big)\Big]^2} \\
&= \frac{\exp\left(-\int p(t)\,dt\right)}{y_1^2(t)}.
\end{aligned}$$

Hence,

$$v(t) = \int w(t)\,dt = \int \frac{\exp\left(-\int p(t)\,dt\right)}{y_1^2(t)}\,dt,$$

where we have set $C = 0$ since we need only one solution, and

$$y_2(t) = y_1(t)v(t) = y_1(t)\int \frac{\exp\left(-\int p(t)\,dt\right)}{y_1^2(t)}\,dt.$$

is a second solution of (6). Since $v(t)$ is obtained by integrating a nonzero function, it cannot be a constant function; it then follows that $y_1(t)$ and $y_2(t)$ are linearly independent.

2.4. **Homogeneous Linear Equations with Constant Coefficients.** Consider now the general homogeneous second-order linear differential equation with constant coefficients,

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0. \tag{7}$$

The trick to solving this equation is to guess that solutions may be of the form $y(t) = e^{rt}$. In that case, we have $y'(t) = re^{rt}$ and $y''(t) = r^2e^{rt}$; plugging these into (7) gives

$$(ar^2 + br + c)e^{rt} = 0.$$

Since e^{rt} is never zero, we can divide through by it, leaving us with a quadratic equation for r:

$$ar^2 + br + c = 0.$$

[9]We only need one, nonzero, solution, so we set $C = 1$.

The quadratic formula then yields

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Depending on the value of $b^2 - 4ac$, we may obtain two real roots, two complex roots, or one real root.

2.4.1. *Two Real Roots* ($b^2 - 4ac > 0$). In this case, we have the two linearly independent solutions

$$\begin{aligned} y_1(t) &= e^{r_1 t} \\ y_2(t) &= e^{r_2 t}, \end{aligned}$$

where

$$\begin{aligned} r_1 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ r_2 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

The general solution is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

2.4.2. *Two Complex Roots* ($b^2 - 4ac < 0$). In this case, we have the two linearly independent solutions

$$\begin{aligned} z_1(t) &= e^{(\lambda + i\mu)t} \\ z_2(t) &= e^{(\lambda - i\mu)t}, \end{aligned}$$

where

$$\begin{aligned} \lambda &= -\frac{b}{2a}, \\ \mu &= \frac{\sqrt{4ac - b^2}}{2a}. \end{aligned}$$

However, these solutions are complex-valued and we only want real solutions. To remedy this problem, suppose that $z(t) = u(t) + iv(t)$ is a complex-valued solution of (7), which is reprinted here:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0.$$

Then we would have:

$$\begin{aligned} a\frac{d^2u}{dt^2} + ia\frac{d^2v}{dt^2} + b\frac{du}{dt} + ib\frac{dv}{dt} + cu + icv = 0 \\ \left[a\frac{d^2u}{dt^2} + b\frac{du}{dt} + cu\right] + i\left[a\frac{d^2v}{dt^2} + b\frac{dv}{dt} + cv\right] = 0. \end{aligned}$$

For the complex-valued left-hand side to equal 0, both its real and imaginary parts would have to equal zero. This means that both u and v are real-valued solutions of (7).

In other words, if $z(t)$ is a complex-valued solution of (7), then both $\operatorname{Re}\{z(t)\}$ and $\operatorname{Im}\{z(t)\}$ are real-valued solutions of (7). Since, by Euler's formula,

$$z_1(t) = e^{(\lambda + i\mu)t} = e^{\lambda t + i\mu t} = e^{\lambda t}\big[\cos(\mu t) + i\sin(\mu t)\big],$$

we have

$$\begin{aligned} y_1(t) &= e^{\lambda t}\cos(\mu t) \\ y_2(t) &= e^{\lambda t}\sin(\mu t) \end{aligned}$$

as linearly independent real-valued solutions of (7). According to the discussion of section 2.1, this is sufficient to characterize the general solution of (7). However, it is natural to wonder why we are allowed to discard $z_2(t)$: might it not provide additional solutions? This turns out not to be the case: we have

$$z_2(t) = e^{(\lambda - i\mu)t} = e^{\lambda t - i\mu t} = e^{\lambda t}\big[\cos(\mu t) - i\sin(\mu t)\big],$$

and thus

$$\begin{aligned}\mathrm{Re}\{z_2(t)\} &= y_1(t)\\ \mathrm{Im}\{z_2(t)\} &= -y_2(t).\end{aligned}$$

Thus, the components of $z_2(t)$ are just scalar multiples of the components of $z_1(t)$, and are already accounted for in the general solution,

$$y(t) = e^{\lambda t}\big[C_1\cos(\mu t) + C_2\sin(\mu t)\big].$$

2.4.3. *One Real Root* ($b^2 - 4ac = 0$). In this case, we have one solution $y_1(t) = e^{r_1 t}$, where $r_1 = -b/(2a)$, and can use two different methods to find a second.

The first method is reduction of order. To use the results of section 2.3, we must first divide through by a in order to put (7) in standard form:

$$\frac{d^2y}{dt^2} + \left(\frac{b}{a}\right)\frac{dy}{dt} + \left(\frac{c}{a}\right)y = 0.$$

We now have $p(t) = b/a$, so from our work in section 2.3 we have:

$$\begin{aligned}y_2(t) &= y_1(t)\int \frac{\exp\left(-\int p(t)\,dt\right)}{y_1^2(t)}\,dt\\ &= e^{r_1 t}\int \frac{\exp\left(-\int b/a\,dt\right)}{\left[e^{-b/(2a)}\right]^2}\,dt\\ &= e^{r_1 t}\int \frac{e^{-b/a}}{e^{-b/a}}\,dt\\ &= e^{r_1 t}\int dt\\ &= te^{r_1 t}.\end{aligned}$$

We now have two linearly independent solutions $y_1(t)$ and $y_2(t)$.

To use the second method, we can define the operator L as

(8) $$L\big[y\big] = a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy.$$

Plugging $y = e^{rt}$ into (8) gives us

$$L\big[e^{rt}\big] = (ar^2 + br + c)e^{rt},$$

and since $r = r_1$ is a root of the quadratic equation $ar^2 + br + c$, it follows that

$$L\big[e^{r_1 t}\big] = 0,$$

which shows that $y_1(t) = e^{r_1 t}$ is a solution of (7). However, we already knew this. To find a second solution, first notice that we can use the equality of mixed partials to conclude:

$$\begin{aligned}\frac{\partial}{\partial r}L\big[y\big] &= \frac{\partial}{\partial r}\left[a\frac{d^2y}{dt^2} + b\frac{\partial y}{\partial t} + cy\right]\\ &= a\frac{\partial^3 y}{\partial r\,\partial t^2} + b\frac{\partial^2 y}{\partial r\,\partial t} + c\frac{\partial y}{\partial r}\end{aligned}$$

$$\begin{aligned}
&= a\frac{\partial^3 y}{\partial t^2 \partial r} + b\frac{\partial^2 y}{\partial t \partial r} + c\frac{\partial y}{\partial r} \\
&= a\frac{\partial^2}{\partial t^2}\left[\frac{\partial y}{\partial r}\right] + b\frac{\partial}{\partial t}\left[\frac{\partial y}{\partial r}\right] + c\frac{\partial y}{\partial r} \\
&= L\left[\frac{\partial y}{\partial r}\right].
\end{aligned}$$

Now observe that because $r = r_1$ is a double root of $ar^2 + br + c$, we can factor the latter expression to $a(r - r_1)^2$. Thus,

$$L\big[e^{rt}\big] = a(r - r_1)^2 e^{rt}.$$

Differentiating both sides of this relation with respect to r gives:

$$\begin{aligned}
L\left[\frac{\partial}{\partial r}e^{rt}\right] &= a(r - r_1)^2\frac{\partial}{\partial r}\left[e^{rt}\right] + ae^{rt}\frac{\partial}{\partial r}\left[(r - r_1)^2\right] \\
L\big[te^{rt}\big] &= at(r - r_1)^2 e^{rt} + 2a(r - r_1)e^{rt}.
\end{aligned}$$

Finally, by letting $r = r_1$, we see that

$$L\big[te^{r_1 t}\big] = 0,$$

which means that $y_2(t) = te^{r_1 t}$ is a second solution of (7). To be honest, there is no reason in particular why one might be inspired to differentiate the general form of a solution with respect to r. However, it turns out that this technique works in many other cases as well, and it is often much, much easier than the method of reduction of order.

Either way, the general solution is

$$y(t) = (C_1 + C_2 t)e^{r_1 t}.$$

2.5. **Variation of Parameters.** Consider the nonhomogeneous second-order linear differential equation

$$\frac{d^2 y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \tag{9}$$

and the corresponding homogeneous equation

$$\frac{d^2 y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0. \tag{10}$$

Suppose that we have two linearly independent solutions $y_1(t)$ and $y_2(t)$ of (10). Then the general solution of (10) is

$$y(t) = C_1 y_1(t) + C_2 y_2(t).$$

Suppose that we now wish to find a particular solution $\psi(t)$ to (9). We may do this by supposing such a solution is of the form

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t); \tag{11}$$

that is, by letting the parameters C_1 and C_2 vary with time. Notice that we have replaced the problem of finding an appropriate single function $\psi(t)$ by the problem of finding two appropriate functions $u_1(t)$ and $u_2(t)$; in a sense, we have added an additional degree of freedom to our problem. As such, we are justified in adding an additional constraint on $u_1(t)$ and $u_2(t)$ without making it so that the solution set of our problem is empty.

We now compute

$$\frac{d\psi}{dt} = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t).$$

Note that if

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0, \tag{12}$$

then the second derivative of $\psi(t)$ will have no second-order derivatives of $u_1(t)$ or $u_2(t)$. In accordance with our discussion above, we arbitrarily assume that (12) holds. In this case,

$$\frac{d\psi}{dt} = u_1(t)y_1'(t) + u_2(t)y_2'(t),$$

and

$$\frac{d^2\psi}{dt^2} = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t).$$

Plugging these expressions into (9) gives

$$\begin{aligned} &u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t) \\ &+ p(t)u_1(t)y_1'(t) + p(t)u_2(t)y_2'(t) + q(t)u_1(t)y_1(t) + q(t)u_2(t)y_2(t) = g(t), \end{aligned}$$

which may be factored to

$$\begin{aligned} &u_1(t)\Big[y_1''(t) + p(t)y_1'(t) + q(t)y_1(t)\Big] + u_2(t)\Big[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)\Big] \\ &+ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \end{aligned}$$

Because $y_1(t)$ and $y_2(t)$ are solutions of (10), the first two terms cancel and we are left with

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t).$$

Surprisingly, when we combine this relation with (12), we find that we can solve algebraically for $u_1'(t)$ and $u_2'(t)$. Doing so gives:

$$\begin{aligned} u_1'(t) &= \frac{-g(t)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}, \\ u_2'(t) &= \frac{g(t)y_1(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}. \end{aligned}$$

Hence, a particular solution of (9) is given by:

$$\psi(t) = y_1(t)\int \frac{-g(t)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}\,dt + y_2(t)\int \frac{g(t)y_1(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}\,dt.$$

The method of variation of parameters will solve any nonhomogeneous second-order linear differential equation, assuming that the general solution to the corresponding homogeneous equation has already been computed, but it is often impractical because of the difficulty of the integrals it requires.

2.6. **Method of Judicious Guessing.** [10]

In this section we will develop a method for solving any nonhomogeneous second-order linear differential equation of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = \sum_{i=1}^{N} p_i(t)e^{\lambda_i t}q_i(t),$$

where each of the $p_i(t)$ is a polynomial in t and each of the $q_i(t)$ is either $\cos(\mu_i t)$ or $\sin(\mu_i t)$.

[10] This is also known as the method of undetermined coefficients.

2.6.1. *Polynomial.* We will start with the simple case where the right-hand side is just a polynomial:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n. \tag{13}$$

Notice that if y is a polynomial of degree n, then the left-hand side will also be a polynomial of degree n. Thus it is reasonable to suppose that

$$y(t) = A_0 + A_1t + A_2t^2 + \cdots + A_{n-2}t^{n-2} + A_{n-1}t^{n-1} + A_nt^n. \tag{14}$$

Then,

$$\begin{aligned} y'(t) &= A_1 + 2A_2t + 3A_3t^2 + \cdots + (n-1)A_{n-1}t^{n-2} + nA_nt^{n-1} \\ y''(t) &= 2A_2 + 6A_3t + 12A_4t^2 + \cdots + n(n-1)A_nt^{n-2}, \end{aligned}$$

and plugging these into (13) gives

$$\begin{aligned} &2aA_2 + 6aA_3t + 12aA_4t^2 + \cdots + n(n-1)aA_nt^{n-2} \\ &+ bA_1 + 2bA_2t + 3bA_3t^2 + \cdots + (n-1)bA_{n-1}t^{n-2} + nbA_nt^{n-1} \\ &+ cA_0 + cA_1t + cA_2t^2 + \cdots + cA_{n-2}t^{n-2} + cA_{n-1}t^{n-1} + cA_nt^n \\ &= a_0 + a_1t + a_2t^2 + \cdots + a_nt^n. \end{aligned}$$

We can group like terms on the left-hand side to obtain:

$$\begin{aligned} &\big[cA_0 + bA_1 + 2aA_2\big] + \big[cA_1 + 2bA_2 + 6aA_3\big]t + \big[cA_2 + 3bA_3 + 12aA_4\big]t^2 + \cdots \\ &+ \big[cA_{n-2} + (n-1)bA_{n-1} + n(n-1)aA_n\big]t^{n-2} + \big[cA_{n-1} + nbA_n\big]t^{n-1} + cA_nt^n \\ &= a_0 + a_1t + a_2t^2 + \cdots + a_nt^n. \end{aligned}$$

Setting equivalent powers of t equal and solving for the coefficients $A_0, A_1, \ldots, A_n$, we find that:

$$\begin{aligned} A_n &= \frac{a_n}{c} \\ A_{n-1} &= \frac{a_{n-1} - nbA_n}{c} \\ A_{n-2} &= \frac{a_{n-2} - (n-1)bA_{n-1} - n(n-1)aA_n}{c} \\ &\vdots \\ A_2 &= \frac{a_2 - 3bA_3 - 12aA_4}{c} \\ A_1 &= \frac{a_1 - 2bA_2 - 6aA_3}{c} \\ A_0 &= \frac{a_0 - bA_1 - 2aA_2}{c}. \end{aligned}$$

If $c \neq 0$, then we have a particular solution of (13) given by (14). On the other hand, if $c = 0$, we may apply the reasoning of section 2.2 and let $v = dy/dt$. On doing so, (13) becomes

$$a\frac{dv}{dt} + bv = a_0 + a_1t + a_2t^2 + \cdots + a_nt^n. \tag{15}$$

As before, we assume that

$$v(t) = A_0 + A_1t + \cdots + A_nt^n. \tag{16}$$

It would be easy to solve for the coefficients as we did before, but to simplify our work note that (15) can be obtained from (13) by making the replacements

$$\begin{aligned} y &\to v \\ a &\to 0 \\ b &\to a \\ c &\to b. \end{aligned}$$

Thus we can re-use our work from earlier:

$$\begin{aligned} A_n &= \frac{a_n}{b} \\ A_{n-1} &= \frac{a_{n-1} - naA_n}{b} \\ A_{n-2} &= \frac{a_{n-2} - (n-1)aA_{n-1}}{b} \\ &\vdots \\ A_2 &= \frac{a_2 - 3aA_3}{b} \\ A_1 &= \frac{a_1 - 2aA_2}{b} \\ A_0 &= \frac{a_0 - aA_1}{b}. \end{aligned}$$

This gives us the coefficients for $v(t)$; to obtain $y(t)$ we simply integrate (16):

$$y(t) = A_0 t + \frac{A_1 t^2}{2} + \frac{A_2 t^3}{3} + \cdots + \frac{a_n t^{n+1}}{n+1}.$$

We have set $C = 0$ because we only need one particular solution. (Any constant would be a solution of the homogeneous equation corresponding to (13) if $c = 0$.) Of course, here we run into trouble if $b = 0$. Now, we could re-use our work above, but note that if b and c are both zero, then solving (13) is trivial: we simply integrate twice and divide by a. This gives

$$y(t) = \frac{1}{a}\left[\frac{a_0 t^2}{2} + \frac{a_1 t^3}{6} + \frac{a_2 t^4}{12} + \cdots + \frac{a_n t^{n+2}}{(n+1)(n+2)}\right].$$

Looking at the results of this section, we may conclude that particular solutions of (13) occur in the forms:

$$\begin{aligned} &A_0 + A_1 t + \cdots + A_n t^n, && \text{if } c \neq 0, \\ t&\Big[A_0 + A_1 t + \cdots + A_n t^n\Big], && \text{if } c = 0 \text{ but } b \neq 0, \\ t^2&\Big[A_0 + A_1 t + \cdots + A_n t^n\Big], && \text{if } c = b = 0. \end{aligned}$$

2.6.2. *Polynomial with Exponential.* We now consider equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = (a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n)e^{\gamma t}. \tag{17}$$

To reduce (17) to (13), we assume that $y(t) = v(t)e^{\gamma t}$. From this, we have

$$\frac{dy}{dt} = \left[\frac{dv}{dt} + \gamma v\right] e^{\gamma t}$$

$$\frac{d^2y}{dt^2} = \left[\frac{d^2v}{dt^2} + 2\gamma\frac{dv}{dt} + \gamma^2 v\right] e^{\gamma t},$$

and from (17) it follows that

$$\left[a\frac{d^2v}{dt^2} + 2a\gamma\frac{dv}{dt} + a\gamma^2 v + b\frac{dv}{dt} + b\gamma v + cv\right] e^{\gamma t} = (a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n)e^{\gamma t}$$

$$a\frac{d^2v}{dt^2} + \left[2a\gamma + b\right]\frac{dv}{dt} + \left[a\gamma^2 + b\gamma + c\right]v = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n.$$

This equation can now be solved for $v(t)$ according to the techniques of the previous section; then, $y(t) = v(t)e^{\gamma t}$. From the summary given in the previous section, we may conclude that particular solutions of (17) occur in the forms:

$$\begin{aligned} &\left[A_0 + A_1 t + \cdots + A_n t^n\right]e^{\gamma t}, && \text{if } a\gamma^2 + b\gamma + c \neq 0, \\ t&\left[A_0 + A_1 t + \cdots + A_n t^n\right]e^{\gamma t}, && \text{if } a\gamma^2 + b\gamma + c = 0 \text{ but } 2a\gamma + b \neq 0, \\ t^2&\left[A_0 + A_1 t + \cdots + A_n t^n\right]e^{\gamma t}, && \text{if } a\gamma^2 + b\gamma + c = 2a\gamma + b = 0. \end{aligned}$$

However, notice that the expressions appearing above have an interesting relationship to the homogeneous equation corresponding to (17),

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0. \tag{18}$$

In particular, as we saw in section 2.4, if $a\gamma^2 + b\gamma + c = 0$, then $y = e^{\gamma t}$ is a solution of (18). If we also have $2a\gamma + b = 0$, or equivalently $\gamma = -b/(2a)$, then γ is a double root of $a\gamma^2 + b\gamma + c = 0$. In this case, both $y = e^{\gamma t}$ and $y = te^{\gamma t}$ are solutions of (18). So we may conclude that particular solutions of (17) occur in the forms:

$$\begin{aligned} &\left[A_0 + A_1 t + \cdots + A_n t^n\right]e^{\gamma t}, && \text{if } y = e^{\gamma t} \text{ does not solve (18)}, \\ t&\left[A_0 + A_1 t + \cdots + A_n t^n\right]e^{\gamma t}, && \text{if } y = e^{\gamma t} \text{ solves (18) but } y = te^{\gamma t} \text{ does not}, \\ t^2&\left[A_0 + A_1 t + \cdots + A_n t^n\right]e^{\gamma t}, && \text{if both } y = e^{\gamma t} \text{ and } y = te^{\gamma t} \text{ solve (18)}. \end{aligned}$$

2.6.3. *Polynomial with Trigonometric Function.* We now consider equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + c\frac{dy}{dt} = (a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n)e^{\lambda t}\cos(\mu t). \tag{19}$$

To solve this equation, note that

$$\operatorname{Re}\left\{(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n)e^{(\lambda + i\mu)t}\right\} = (a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n)e^{\lambda t}\cos(\mu t).$$

According to our discussion in section 2.4.2, if we can find a complex-valued solution $z(t)$ of

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + c\frac{dy}{dt} = (a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n)e^{(\lambda + i\mu)t},$$

then $\operatorname{Re}\{z(t)\}$ will be a real-valued solution of (19). If we let $\gamma = \lambda + i\mu$, then the above equation may be solved using the techniques of the previous section, although the coefficients $A_0, A_1, \ldots, A_n$ will now be complex-valued. Taking the real part of the resulting function gives a particular solution to (19).

Tracing the calculations of this method allows us to conclude that particular solutions of (19) occur in the forms:

$$\Big[A_0 + A_1 t + \cdots + A_n t^n\Big] e^{\lambda t}\cos(\mu t) + \Big[B_0 + B_1 t + \cdots + B_n t^n\Big] e^{\lambda t}\sin(\mu t),$$
$$\text{if } y = e^{\lambda t}\cos(\mu t) \text{ does not solve (18)},$$
$$t\Big[A_0 + A_1 t + \cdots + A_n t^n\Big] e^{\lambda t}\cos(\mu t) + t\Big[B_0 + B_1 t + \cdots + B_n t^n\Big] e^{\lambda t}\sin(\mu t),$$
$$\text{if } y = e^{\lambda t}\cos(\mu t) \text{ solves (18) but } y = te^{\lambda t}\cos(\mu t) \text{ does not},$$
$$t^2\Big[A_0 + A_1 t + \cdots + A_n t^n\Big] e^{\lambda t}\cos(\mu t) + t^2\Big[B_0 + B_1 t + \cdots + B_n t^n\Big] e^{\lambda t}\sin(\mu t),$$
$$\text{if both } y = e^{\lambda t}\cos(\mu t) \text{ and } y = te^{\lambda t}\cos(\mu t) \text{ solve (18)}.$$

Of course, by replacing cos with sin and Re with Im, we may instead obtain solutions of equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + c\frac{dy}{dt} = (a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n)e^{\lambda t}\sin(\mu t).$$

2.6.4. *General Equation.* Observe that our work in section 2.6.3 allows us to solve any nonhomogeneous second-order linear differential equation of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = p_i(t)e^{\lambda_i t}q_i(t), \tag{20}$$

where $p_i(t)$ is a polynomial in t and $q_i(t)$ is either $\cos(\mu_i t)$ or $\sin(\mu_i t)$. (We can recover the equations of sections 2.6.2 and 2.6.1 by setting $\mu = 0$ and $\lambda = 0$, respectively.) The procedure for solving the general equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = \sum_{i=1}^{N} p_i(t)e^{\lambda_i t}q_i(t) \tag{21}$$

presented at the beginning of section 2.6 is simple, if tedious for large N. For each i from 1 to N, find a particular solution $y_i(t)$ to (20). Then, it is easy to see that a particular solution to (21) is given by

$$y(t) = \sum_{i=1}^{N} y_i(t).$$

It should also be noted that instead of following the procedures outlined in sections 2.6.1–2.6.4, we can instead use the summaries given at the end of each of those sections. For instance, if the right-hand side of (21) is $t^3e^{2t}\sin(3t) + t^2$, and both $e^{2t}\sin(3t)$ and $te^{2t}\sin(3t)$ are solutions of the corresponding homogeneous equation, we may assume that $y(t)$ is of the form

$$y(t) = t^2\Big[A_0 + A_1 t + A_2 t^2 + A_3 t^3\Big]e^{2t}\cos(3t) + t^2\Big[B_0 + B_1 t + B_2 t^2 + B_3 t^3\Big]e^{2t}\sin(3t) + C_0 + C_1 t + C_2 t^2,$$

plug this into (21), and solve for the parameters A_i, B_i, C_i by equating the coefficients of linearly independent terms. Which of these two approaches is easier depends on the problem at hand.

2.7. **Power Series Solutions.** Consider the homogeneous second-order linear differential equation

$$P(t)\frac{d^2y}{dt^2} + Q(t)\frac{dy}{dt} + R(t)y = 0, \tag{22}$$

where $P(t)$, $Q(t)$, and $R(t)$ have series expansions given by:

$$P(t) = \sum_{n=0}^{\infty} p_n t^n, \qquad Q(t) = \sum_{n=0}^{\infty} q_n t^n, \qquad R(t) = \sum_{n=0}^{\infty} r_n t^n.$$

It is reasonable to suppose that solutions will be of the same form; that is,

$$y(t) = \sum_{n=0}^{\infty} a_n t^n.$$

Differentiating, we find that

$$\begin{aligned} \frac{dy}{dt} &= \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n \\ \frac{d^2y}{dt^2} &= \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n, \end{aligned}$$

Substituting these series into (22), we obtain:

$$\sum_{n=0}^{\infty} p_n t^n \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n + \sum_{n=0}^{\infty} q_n t^n \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n + \sum_{n=0}^{\infty} r_n t^n \sum_{n=0}^{\infty} a_n t^n = 0.$$

Now we may use the following elementary property of power series:

$$\sum_{n=0}^{\infty} a_n t^n \sum_{n=0}^{\infty} b_n t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} t^n.$$

Doing so yields:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} (k+2)(k+1) p_{n-k} a_{k+2} t^n + \sum_{n=0}^{\infty} \sum_{k=0}^{n} (k+1) q_{n-k} a_{k+1} t^n + \sum_{n=0}^{\infty} \sum_{k=0}^{n} r_{n-k} a_k t^n = 0.$$

Combining the summations, we find that

$$\sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \left[(k+2)(k+1) p_{n-k} a_{k+2} + (k+1) q_{n-k} a_{k+1} + r_{n-k} a_k \right] \right\} t^n = 0,$$

which implies that for every $n \geq 0$ we have

$$\sum_{k=0}^{n} \left[(k+2)(k+1) p_{n-k} a_{k+2} + (k+1) q_{n-k} a_{k+1} + r_{n-k} a_k \right] = 0. \tag{23}$$

For the moment, assume that $p_0 \neq 0$. In that case we can solve for a_{n+2}, as follows. Pulling out the last term of the summation, for which $k = n$, we obtain

$$\begin{aligned} &\sum_{k=0}^{n-1} \left[(k+2)(k+1) p_{n-k} a_{k+2} + (k+1) q_{n-k} a_{k+1} + r_{n-k} a_k \right] \\ &+ (n+2)(n+1) p_0 a_{n+2} + (n+1) q_0 a_{n+1} + r_0 a_n = 0, \end{aligned}$$

and solving for a_{n+2} gives

$$a_{n+2} = - \left\{ \frac{\sum_{k=0}^{n-1} \left[(k+2)(k+1) p_{n-k} a_{k+2} + (k+1) q_{n-k} a_{k+1} + r_{n-k} a_k \right] + (n+1) q_0 a_{n+1} + r_0 a_n}{(n+2)(n+1) p_0} \right\}.$$

The above equation holds for $n \geq 0$, and it determines the value of a_{n+2} in terms of $a_0, a_1, \ldots, a_{n+1}$. Thus, a_0 and a_1 are arbitrary, but once their values are specified, the values of $a_2, a_3, \ldots$ follow from the above equation. Any pair of values for a_0 and a_1, then, will determine a solution

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

of (22). Typically, we want two linearly independent solutions, and typically the easiest way to obtain them is to use $a_0 = 1$, $a_1 = 0$ for the first solution and $a_0 = 0$, $a_1 = 1$ for the second solution.

Notice from the series for $P(t)$ that $p_0 = P(0)$. Thus, the method of power series solution will solve any differential equation (22) for which $P(0) \neq 0$. In particular, it will solve any equation of the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0, \tag{24}$$

where $p(t)$ and $q(t)$ may be expressed as power series. Of course, we may transform (22) into (24) by dividing through by $P(t)$, but the resulting $p(t) = Q(t)/P(t)$ and $q(t) = R(t)/P(t)$ will not necessarily be expressible as power series if $P(0) = 0$. In the next section, we will see how to solve (24) when $p(t)$ and $q(t)$ cannot be expressed as power series but can still be expressed as more general series.

We should also note that it is entirely possible to use the above method with power series about any point (say $t = t_0$), not just $t = 0$. Simply expand $P(t)$, $Q(t)$, and $R(t)$ as power series about t_0, assume that

$$y(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n,$$

and follow the same procedure as outlined above. Expanding about a different point may be useful in order to avoid a singularity: for instance, suppose that $P(0) = 0$ and we consequently cannot find a solution $y(t)$ as a power series about $t = 0$. In that case, taking our power series to be centered at a different point would allow us to find a solution. Unfortunately, it is likely that the resulting series for $y(t)$ would not converge near $t = 0$, so to determine the behavior of $y(t)$ near $t = 0$ we would have to use a different technique.

Finally, note that if our power series are centered at $t = t_0$ and we have the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$, then it follows easily that $a_0 = y_0$ and $a_1 = y_0'$. On the other hand, if our initial conditions are given at a different point than that about which our power series are centered, we will have to solve for the arbitrary constants as usual.

2.8. **Method of Frobenius.** As we discussed in section 2.7, we can solve any differential equation of the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0, \tag{25}$$

if $p(t)$ and $q(t)$ can be expressed as power series. However, if $p(t)$ and $q(t)$ can be expressed as series of the form[11]:

$$p(t) = \frac{p_0}{t} + p_1 + p_2 t + p_3 t^2 + \cdots = \sum_{n=0}^{\infty} p_n t^{n-1}$$

$$q(t) = \frac{q_0}{t^2} + \frac{q_1}{t} + q_2 + q_3 t + \cdots = \sum_{n=0}^{\infty} q_n t^{n-2},$$

[11] To find a series expansion of this type for $p(t)$, divide the power series for $tp(t)$ by t. For $q(t)$, divide the power series for $t^2q(t)$ by t^2.

then we may still be able to find a series solution, using what is called the method of Frobenius. The essence of this method is to account for the singularity at $t = 0$ by multiplying our guess for $y(t)$ by an arbitrary factor of t^r.[12] That is, we assume

$$y(t) = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+r}.$$

With a solution of this form, we may run into trouble with negative t—for instance, obtaining terms of the form $(-1)^{1/2}$. However, restricting our analysis to $t > 0$ is not really a problem since the differential equation is not even defined for $t = 0$.[13]

Differentiating, we find:

$$\frac{dy}{dt} = \sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1}$$

$$\frac{d^2y}{dt^2} = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-2}.$$

Plugging the series for $p(t)$, $q(t)$, $y(t)$, $y'(t)$, and $y''(t)$ into (25) gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-2} + \sum_{n=0}^{\infty} p_n t^{n-1} \sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1} + \sum_{n=0}^{\infty} q_n t^{n-2} \sum_{n=0}^{\infty} a_n t^{n+r} = 0.$$

Using the same property of multiplication of power series that we used in section 2.7, we obtain

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-2} + \sum_{n=0}^{\infty}\sum_{k=0}^{n} p_{n-k}(k+r)a_k t^{n+r-2} + \sum_{n=0}^{\infty}\sum_{k=0}^{n} q_{n-k} a_k t^{t+r-2} = 0.$$

Grouping then gives

$$\sum_{n=0}^{\infty}\left\{(n+r)(n+r-1)a_n + \sum_{k=0}^{n}\Big[p_{n-k}(k+r)a_k + q_{n-k}a_k\Big]\right\} t^{n+r-2} = 0.$$

Now we will pull out the last term of the inner summation:

$$\sum_{n=0}^{\infty}\left\{\Big[(n+r)(n+r-1) + p_0(n+r) + q_0\Big]a_n + \sum_{k=0}^{n-1}\Big[p_{n-k}(k+r)a_k + q_{n-k}a_k\Big]\right\} t^{n+r-2} = 0.$$

Finally, for convenience we introduce the notation $F(r) = r(r-1) + p_0 r + q_0$, which gives:

$$\sum_{n=0}^{\infty}\left\{F(n+r)a_n + \sum_{k=0}^{n-1}\Big[p_{n-k}(k+r)a_k + q_{n-k}a_k\Big]\right\} t^{n+r-2} = 0. \tag{26}$$

Setting the coefficient of each power of t to zero, we find that for every $n \geq 0$

$$F(n+r)a_n = -\sum_{k=0}^{n-1}\Big[p_{n-k}(k+r)a_k + q_{n-k}a_k\Big]. \tag{27}$$

For $n = 0$ we have

$$F(r)a_0 = 0.$$

[12] This is inspired by the fact that fractional powers of t cannot, in general, be expressed as power series centered at $t = 0$.

[13] To find solutions for $t < 0$ instead, simply use the change of variables $x = -t$.

We may assume that $a_0 \neq 0$, because if $a_0 = 0$ then we can just relabel a_1 as a_0, a_2 as a_1, and so on, and increase r by 1, to give a solution with $a_0 \neq 0$. Thus we have

$$F(r) = 0.$$

This is called the **indicial equation** of (25), and it has roots given by

$$r_1 = \frac{-(p_0 - 1) + \sqrt{(p_0 - 1)^2 - 4q_0}}{2}, \qquad r_2 = \frac{-(p_0 - 1) - \sqrt{(p_0 - 1)^2 - 4q_0}}{2}. \tag{28}$$

From here, generally speaking, there are three cases to consider. The first is when $r_1 \neq r_2$ and $r_1 - r_2$ is not an integer; the second is when $r_1 = r_2$; and the third is when $r_1 \neq r_2$ and $r_1 - r_2$ is an integer.

2.8.1. *Roots Not Differing by an Integer.* Recall that (27) gives us the indicial equation in the case of $n = 0$. For $n \geq 1$, we may solve it for a_n to find:

$$a_n(r) = -\left\{ \frac{\displaystyle\sum_{k=0}^{n-1} \Big[p_{n-k}(k + r) + q_{n-k} \Big] a_k}{F(n + r)} \right\}. \tag{29}$$

We are assured that $F(n + r)$ is never zero if r is a root of $F(r)$, because in this case the two roots of $F(r)$ do not differ by an integer. Also, we have introduced the notation $a_n(r)$ to emphasize that the value of each a_n depends on the choice of r.

The value is a_0 is arbitrary; we can select any nonzero value for it. Once we do, however, the values of $a_1(r), a_2(r), \ldots$ are determined by (29). Two linearly independent solutions of (25) are then given by:

$$y_1(t) = t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n$$
$$y_2(t) = t^{r_2} \sum_{n=0}^{\infty} a_n(r_2) t^n.$$

If r_1 and r_2 are complex, then in theory two linearly independent solutions are given by:

$$y_1(t) = \mathrm{Re} \left\{ t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n \right\}$$
$$y_2(t) = \mathrm{Im} \left\{ t^{r_1} \sum_{n=0}^{\infty} a_n(r_2) t^n \right\}.$$

Practically speaking, these computations may be extremely difficult.

2.8.2. *Equal Roots.* As we did in section 2.4.3, we define the operator L as

$$L\big[y\big] = \frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t) y.$$

Our derivation of (26) illustrates that

$$L\left[t^r \sum_{n=0}^{\infty} a_n t^n \right] = \sum_{n=0}^{\infty} \left\{ F(n + r) a_n + \sum_{k=0}^{n-1} \Big[p_{n-k}(k + r) a_k + q_{n-k} a_k \Big] \right\} t^{n+r-2}.$$

Now, if we let a_0 be an arbitrary, nonzero constant (independent of the value of r) and define the coefficients $a_1, a_2, \ldots$ according to (29), then this equation reduces to

$$L\left[t^r \sum_{n=0}^{\infty} a_n(r) t^n\right] = F(r) a_0 t^{r-2}.$$

Since $r = r_1$ is a root of $F(r)$, we see that

$$L\left[t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n\right] = 0,$$

which shows that $y_1(t) = t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n$ is a solution of (25). However, we already knew this. To find a second solution, first notice that just as in section 2.4.3 we can use the equality of mixed partials to conclude

$$\frac{\partial}{\partial r} L[y] = L\left[\frac{\partial y}{\partial r}\right].$$

Now observe that because $r = r_1$ is a double root of $F(r) = r(r-1) + p_0 r + q_0$, we can factor the latter expression to $(r - r_1)^2$. Thus,

$$L\left[t^r \sum_{n=0}^{\infty} a_n(r) t^n\right] = (r - r_1)^2 a_0 t^{r-2}.$$

Differentiating both sides of this relation with respect to r gives:

$$L\left[\frac{\partial}{\partial r} t^r \sum_{n=0}^{\infty} a_n(r) t^n\right] = \frac{\partial}{\partial r}\left[(r - r_1)^2 a_0 t^{r-2}\right]$$

$$L\left[(\ln t) t^r \sum_{n=0}^{\infty} a_n(r) + t^r \sum_{n=0}^{\infty} a_n'(r)\right] = 2(r - r_1) a_0 t^{r-2} + (r - r_1)^2 a_0 (\ln r) t^{r-2}.$$

Finally, by letting $r = r_1$, we see that

$$L\left[(\ln t) t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) + t^{r_1} \sum_{n=0}^{\infty} a_n'(r_1)\right] = 0,$$

which means that

$$y_2(t) = (\ln t) t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) + t^{r_1} \sum_{n=0}^{\infty} a_n'(r_1)$$

is a second solution of (25).

2.8.3. *Roots Differing by an Integer.* As we have written them in (28), r_1 is greater than r_2. Hence, for all $n \geq 1$, it follows that $F(n + r_1) \neq 0$. Thus we can obtain one solution

$$y_1(t) = t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n$$

via the methods of section 2.8.1. However, if we try to obtain a second solution of the form $t^{r_2} \sum_{n=0}^{\infty} a_n(r_2) t^n$, we will run into trouble because $F(r_2 + n) = 0$ for $n = r_1 - r_2$. If we are lucky, then for $n = r_1 - r_2$, we will have

$$\sum_{k=0}^{n-1} \left[p_{n-k}(k + r) a_k + q_{n-k} a_k\right] = 0. \tag{30}$$

In that case, (27) tells us that we can pick any value for a_n. Then we can proceed onwards using the methods of section 2.8.1, and we will obtain a second solution of the form

$$y_2(t) = t^{r_2} \sum_{n=0}^{\infty} a_n(r_2) t^n.$$

On the other hand, if we are not lucky enough to have (30) hold, then we cannot find a solution of this form. Referring to a more advanced text on differential equations would show that in this case there is a second solution of the form

$$y_2(t) = (\ln t) t^{r_2} \sum_{n=0}^{\infty} a_n(r_2) + t^{r_2} \sum_{n=0}^{\infty} a_n'(r_2),$$

where $a_0(r)$, which was previously an arbitrary constant independent of r, must now be equal to $r - r_2$.

2.9. **Laplace Transforms.** In this section we discuss a unique method for solving nonhomogeneous second-order linear differential equations of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t) \tag{31}$$

for which we are given initial conditions

$$y(0) = y_0$$
$$y'(0) = y_0'.$$

If the initial conditions are given at $t = t_0$, then use the transformations $\tau = t - t_0$ and $y(t) = y(\tau + t_0) = z(\tau)$, so that the differential equation becomes

$$a\frac{d^2z}{d\tau^2} + b\frac{dz}{d\tau} + cz = f(\tau + t_0)$$

and the initial conditions become

$$z(0) = y_0$$
$$z'(0) = y_0'.$$

This method depends upon an operator $\mathcal{L}$ called the **Laplace transform**. Every function $f(t)$ has a Laplace transform $F(s)$, which is given by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$$

The reason the Laplace transform is useful in solving differential equations is the following property, which we derive using integration by parts with $u = e^{-st}$, $du = -se^{-st}\, dt$, $dv = f'(t)\, dt$, $v = f(t)$:

$$\mathcal{L}\{y'(t)\} = \int_0^{\infty} e^{-st} y'(t)\, dt = \Big[e^{-st} y(t)\Big]_{t=0}^{\infty} + s\int_0^{\infty} e^{-st} y(t)\, dt = -y(0) + s\mathcal{L}\{y(t)\}.$$

Applying this property twice, we obtain:

$$\mathcal{L}\{y''(t)\} = -y'(0) + s\mathcal{L}\{y'(t)\} = -y'(0) - sy(0) + s^2\mathcal{L}\{y(t)\}.$$

Thus, taking Laplace transforms of both sides of (31) gives us

$$-ay'(0) - asy(0) + as^2Y(s) - by(0) + bsY(s) + cY(s) = F(s)$$
$$\big[as^2 + bs + c\big]Y(s) - ay_0 s - y_0' - by_0 = F(s),$$

and we can then solve algebraically for $Y(s)$:

$$Y(s) = \frac{F(s) + ay_0 s + ay_0' + by_0}{as^2 + bs + c}.$$

If we can find a function $y(t)$ such that $\mathcal{L}\{y(t)\} = Y(s)$, which is called the **inverse Laplace transform** of $Y(s)$ and is denoted $\mathcal{L}^{-1}\{Y(s)\}$, then this function is a solution of the original initial-value problem (31).

Typically, tables of Laplace transforms are used to compute $\mathcal{L}\{f(t)\}$ and $\mathcal{L}^{-1}\{Y(s)\}$.

2.10. **Transformation to System.** Given the general homogeneous second-order linear differential equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0,$$

we can define

$$\mathbf{y}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}.$$

It then follows that

$$\begin{aligned} \mathbf{y}'(t) &= \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix} \\ &= \begin{bmatrix} y'(t) \\ -p(t)y'(t) - q(t)y(t) \end{bmatrix} \\ &= y(t) \begin{bmatrix} 0 \\ -q(t) \end{bmatrix} + y'(t) \begin{bmatrix} 1 \\ -p(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \mathbf{y}(t). \end{aligned}$$

That is, we can transform a homogeneous second-order linear differential equation to a system of two homogeneous first-order linear differential equations.

2.11. **What to Do in General.** The following strategy is advisable:

(1) If the equation is missing a y term, use the strategy of section 2.2.
(2) If the equation is linear and homogeneous and has constant coefficients, use the strategy of section 2.4.
(3) If the equation is linear and homogeneous and has nonconstant coefficients, and you already know one solution, use the strategy of section 2.3 to find another.
(4) If the equation is linear and homogeneous and has nonconstant coefficients that can be expanded as power series, use the strategy of section 2.7.
(5) If the equation is linear and homogeneous and has nonconstant coefficients that cannot be expanded as power series but can be expanded as an appropriate modified power series, use the strategy of section 2.8.
(6) If the equation is linear and nonhomogeneous, has constant coefficients, and has a right-hand side of the appropriate form, use the strategy of section 2.4 to find the general solution of the corresponding homogeneous equation and then use the strategy of section 2.6.
(7) If the equation is linear and nonhomogeneous and has constant coefficients but does not have a right-hand side admitting of the method of judicious guessing, either (i) use the strategy of section 2.4 to find the general solution of the corresponding homogeneous equation and then use the strategy of section 2.5, or (ii) use the strategy of section 2.9.

3. Higher-Order Differential Equations

3.1. **Characterization of Higher-Order Equations.** An n**th-order differential equation** is an equation of the form

$$\frac{d^ny}{dt^n} = f\left(t, y, \frac{dy}{dt}, \ldots, \frac{d^{n-1}y}{dt^{n-1}}\right).$$

If n additional equations of the form

$$\begin{aligned} y(t_0) &= y_0 \\ y'(t_0) &= y'_0 \\ &\vdots \\ y^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned}$$

are also given, then the $n+1$ equations together are called an **initial-value problem**. Every initial-value problem has a unique solution; however, if initial conditions are not provided, the general solution will have n arbitrary constants.[14] An nth-order differential equation of the form

$$\frac{d^ny}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = g(t) \tag{32}$$

is called **linear**. If $g(t) = 0$ then the equation is of the form

$$\frac{d^ny}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = 0 \tag{33}$$

and it is called **homogeneous**; otherwise, it is called **nonhomogeneous**.

A set of n functions $y_1(t), y_2(t), \ldots, y_n(t)$ is called **linearly independent** if there is no set of numbers $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) = 0$$

for all t.[15] The **Wronskian** of n functions $y_1(t), y_2(t), \ldots, y_n(t)$ is the quantity

$$W[y_1, y_2, \ldots, y_n] = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y'_1(t) & y'_2(t) & \cdots & y'_n(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}.$$

The Wronskian of n linearly independent solutions of (33) is nonzero for all t, while the Wronskian of n linearly dependent solutions of (33) solutions of (33) is zero for all t.

If $y_1(t), y_2(t), \ldots, y_n(t)$ are n linearly independent solutions of (33), then the general solution to (33) is

$$y(t) = C_1y_1(t) + C_2y_2(t) + \cdots + C_ny_n(t).^{16}$$

If $y_1(t), y_2(t), \ldots, y_n(t)$ are n linearly independent solutions of (33) and $\psi(t)$ is a solution of (32), then the general solution to (32) is

$$y(t) = C_1y_1(t) + C_2y_2(t) + \cdots + C_ny_n(t) + \psi(t).^{17}$$

[14] This requires that f be sufficiently well-behaved.

[15] Three consequences of this definition are that none of the functions may be the zero function, that none of the functions may be a scalar multiple of another, and that none of the functions may be a linear combination of the others.

[16] This requires that the functions $p_0(t), p_1(t), \ldots, p_n(t)$ are sufficiently well-behaved.

[17] This requires that the functions $p_0(t), p_1(t), \ldots, p_n(t), g(t)$ are sufficiently well-behaved.

3.2. **Direct Reduction of Order.** If y does not appear in an nth-order differential equation, i.e. it is of the form

$$\frac{d^n y}{dt^n} = f\left(t, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \ldots, \frac{d^{n-1} y}{dt^{n-1}}\right),$$

then letting $v = dy/dt$ converts it to an $(n-1)$th-order equation:

$$\frac{d^{n-1} v}{dt^{n-1}} = f\left(t, v, \frac{dv}{dt}, \ldots, \frac{d^{n-2} v}{dt^{n-2}}\right).$$

$n-1$ constants of integration are introduced in the solution of this $(n-1)$th-order equation, and an nth is introduced in recovering y:

$$y(t) = \int v(t)\, dt.$$

In the context of an initial-value problem, the initial conditions

$$\begin{aligned} y'(t_0) &= y_0' \\ y''(t_0) &= y_0'' \\ &\vdots \\ y^{(n-1)}(t_0) &= y_0^{(n-1)} \end{aligned}$$

are used in the solution of the $(n-1)$th-order equation and the initial condition $y(t_0) = y_0$ is used in recovering y.

3.3. **Reduction of Order.** We rewrite (33) as

$$\sum_{n=0}^{N} \left[p_n(t) \frac{d^n y}{dt^n} \right] = 0 \tag{34}$$

by replacing n with N and letting $p_N(t) = 1$. Analogously to our work in section 2.3, suppose that we have one solution $y_1(t)$ of (34) and seek $N-1$ more linearly independent solutions. We assume that any other solutions will be of the form

$$y(t) = y_1(t) v(t).$$

Differentiating, we find that

$$\begin{aligned} \frac{dy}{dt} &= \frac{dy_1}{dt} v + y_1 \frac{dv}{dt} \\ \frac{d^2 y}{dt^2} &= \frac{d^2 y_1}{dt^2} v + 2 \frac{dy_1}{dt} \frac{dv}{dt} + y_1 \frac{d^2 v}{dt^2} \\ \frac{d^3 y}{dt^3} &= \frac{d^3 y_1}{dt^3} v + 3 \frac{d^2 y_1}{dt^2} \frac{dv}{dt} + 3 \frac{dy_1}{dt} \frac{d^2 v}{dt^2} + y_1 \frac{d^3 v}{dt^3}, \end{aligned}$$

and in general

$$\frac{d^n y}{dt^n} = \sum_{k=0}^{n} \left[\binom{n}{k} \frac{d^{n-k} y_1}{dt^{n-k}} \frac{d^k v}{dt^k} \right].$$

Hence, from (34) we have

$$\sum_{n=0}^{N} \left\{ p_n(t) \sum_{k=0}^{n} \left[\binom{n}{k} \frac{d^{n-k} y_1}{dt^{n-k}} \frac{d^k v}{dt^k} \right] \right\} = 0.$$

Pulling out each of the $k = 0$ terms gives us

$$v \sum_{n=0}^{N} \left[p_n(t) \frac{d^{n-k} y_1}{dt^{n-k}} \right] + \sum_{n=0}^{N} \left\{ p_n(t) \sum_{k=1}^{n} \left[\binom{n}{k} \frac{d^{n-k} y_1}{dt^{n-k}} \frac{d^k v}{dt^k} \right] \right\} = 0,$$

and (34) tells us we can cancel the left-hand term. Using the strategy of section 3.2, we let $w = dv/dt$ and obtain

$$\sum_{n=0}^{N} \left\{ p_n(t) \sum_{k=1}^{n} \left[\binom{n}{k} \frac{d^{n-k} y_1}{dt^{n-k}} \frac{d^{k-1} w}{dt^{k-1}} \right] \right\}, \tag{35}$$

which is a homogeneous $(n-1)$th-order linear differential equation for $w(t)$. Upon finding $n-1$ linearly independent solutions $v_1(t), v_2(t), \ldots, v_{n-1}(t)$ of (35), our n linearly independent solutions of (34) are then given by:

$$\begin{aligned} y_1(t) &= y_1(t) \\ y_2(t) &= y_1(t) v_1(t) \\ y_3(t) &= y_1(t) v_2(t) \\ &\vdots \\ y_n(t) &= y_1(t) v_{n-1}(t). \end{aligned}$$

3.4. **Homogeneous Linear Equations with Constant Coefficients.** Consider the general homogeneous nth-order linear differential equation with constant coefficients,

$$c_n \frac{d^n y}{dt^n} + c_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + c_1 \frac{dy}{dt} + c_0 y = 0. \tag{36}$$

As in section 2.4, we assume that solutions are of the form $y(t) = e^{rt}$. In that case, $d^k y/dt^k = r^k e^{rt}$ and we obtain

$$(c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r + c_0) e^{rt} = 0,$$

or equivalently

$$c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r + c_0 = 0.$$

This equation has n roots (not necessarily real and not necessary distinct), which give us the solutions

$$\begin{aligned} y_1(t) &= e^{r_1 t} \\ y_2(t) &= e^{r_2 t} \\ &\vdots \\ y_n(t) &= e^{r_n t}. \end{aligned}$$

We now address the cases of complex roots and repeated roots.

Suppose that one of the roots r_i is complex; that is, $r_i = \lambda + i\mu$ for real numbers λ and μ. It then follows that there is another root r_j such that $r_j = \lambda - i\mu$.[18] We then have two complex-valued solutions of (36):

$$\begin{aligned} y_i(t) &= e^{(\lambda + i\mu)t} \\ y_j(t) &= e^{(\lambda - i\mu)t}. \end{aligned}$$

In this case, we can use the reasoning of section 2.4.2 to replace these complex-valued solutions with the real-valued solutions

$$y_i(t) = e^{\lambda t} \cos(\mu t)$$

[18] This result is called the complex conjugate root theorem and is not too difficult to prove.

$$y_j(t) = e^{\lambda t}\sin(\mu t).$$

Now suppose that one of the roots is repeated m times (that is, it has multiplicity m). Without loss of generality, we will assume that $r_1 = r_2 = \cdots = r_m$. As we did in section 2.4.3, we define the operator L as

$$L\big[y\big] = c_n\frac{d^n y}{dt^n} + c_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + c_1\frac{dy}{dt} + c_0 y,$$

so that

$$L\big[e^{rt}\big] = p(r)e^{rt},$$

where

$$p(r) = c_n r^n + c_{n-1}r^{n-1} + \cdots + c_1 r + c_0.$$

As we know, since r_1 is a root of $p(r)$, we have

$$L\big[e^{r_1 t}\big] = 0,$$

but this is old news. What is more interesting is that since r_1 is root of $p(r)$ with multiplicity m, we can factor $p(r)$ into $(r-r_1)^m q_0(r)$, where $q_0(r)$ is a polynomial of degree $n-m$. Thus we have

$$L\big[e^{rt}\big] = q(r)(r-r_1)^m e^{rt}.$$

Differentiating both sides with respect to r and bringing the partial derivative $\partial/\partial r$ within the operator L gives us

$$\begin{aligned} L\big[te^{rt}\big] &= q_0'(r)(r-r_1)^m e^{rt} + mq_0(r)(r-r_1)^{m-1}e^{rt} + tq_0(r)(r-r_1)^m e^{rt} \\ &= \Big[q_0'(r)(r-r_1) + mq_0(r) + tq(r)(r-r_1)\Big](r-r_1)^{m-1}e^{rt} \\ &= q_1(r)(r-r_1)^{m-1}e^{rt}, \end{aligned}$$

where $q_1(r)$ is another polynomial. Plugging in $r = r_1$ then shows that

$$L\big[te^{r_1 t}\big] = 0,$$

and thus $y_2(t) = te^{r_1 t}$ is another solution of (36). Differentiating a second time will give

$$L\big[t^2 e^{rt}\big] = q_2(r)(r-r_1)^{m-2}e^{rt},$$

which produces $y_3(t) = t^2 e^{rt}$ as a third solution of (36). In general, we can repeat this reasoning until we obtain

$$L\big[t^m e^{rt}\big] = q_m(t)e^{rt},$$

at which point the $(r-r_1)$ term has disappeared and we can no longer be assured that the right-hand side will be zero at $r = r_1$. This is fine, though, because before we reach this point we obtain m solutions in total:

$$\begin{aligned} y_1(t) &= e^{r_1 t} \\ y_2(t) &= te^{r_1 t} \\ y_3(t) &= t^2 e^{r_1 t} \\ &\ \vdots \\ y_m(t) &= t^{m-1}e^{r_1 t}. \end{aligned}$$

3.5. **Variation of Parameters.** Suppose that we have n linearly independent solutions of

$$\frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = 0 \tag{37}$$

denoted $y_1(t), y_2(t), \ldots, y_n(t)$, and suppose furthermore that we would like to find a particular solution $\psi(t)$ to

$$\frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)y = g(t). \tag{38}$$

We will assume that the solution is of the form

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t).$$

Similarly to our work in section 2.5, we have replaced the problem of finding one function $\psi(t)$ with the problem of finding n functions $y_1(t), \ldots, y_n(t)$; in a sense, we have added $n-1$ additional degrees of freedom to our problem and are therefore justified in adding $n-1$ additional constraints on $u_1(t), \ldots, u_n(t)$ without making it so that there are no solutions. Differentiating, we find that

$$\frac{d\psi}{dt} = u_1'(t)y_1(t) + u_2'(t)y_2(t) + \cdots + u_n'(t)y_n(t) + u_1(t)y_1'(t) + u_2(t)y_2'(t) + \cdots + u_n(t)y_n'(t).$$

To avoid second-order derivatives of $u_1(t), \ldots, u_n(t)$ from appearing when we differentiate this expression, we make the arbitrary assumption that

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) + \cdots + u_n'(t)y_n(t) = 0,$$

so that

$$\begin{aligned}
\frac{d\psi}{dt} &= u_1(t)y_1'(t) + u_2(t)y_2'(t) + \cdots + u_n(t)y_n'(t) \\
\frac{d^2\psi}{dt^2} &= u_1'(t)y_1'(t) + u_2'(t)y_2'(t) + \cdots + u_n'(t)y_n'(t) + u_1(t)y_1''(t) + u_2(t)y_2''(t) + \cdots + u_n(t)y_n''(t).
\end{aligned}$$

We now make the assumption that

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) + \cdots + u_n'(t)y_n'(t) = 0,$$

and proceeding in this manner we find that

$$\begin{aligned}
\frac{d\psi}{dt} &= u_1(t)y_1'(t) + u_2(t)y_2'(t) + \cdots + u_n(t)y_n'(t) \\
\frac{d^2\psi}{dt^2} &= u_1(t)y_1''(t) + u_2(t)y_2''(t) + \cdots + u_n(t)y_n''(t) \\
&\vdots \\
\frac{d^{n-1}\psi}{dt^{n-1}} &= u_1(t)y_1^{(n-1)}(t) + u_2(t)y_2^{(n-1)}(t) + \cdots + u_n(t)y_n^{(n-1)}(t),
\end{aligned}$$

assuming that

$$\begin{aligned}
u_1'(t)y_1(t) + u_2'(t)y_2(t) + \cdots + u_n'(t)y_n(t) &= 0 \\
u_1'(t)y_1'(t) + u_2'(t)y_2'(t) + \cdots + u_n'(t)y_n'(t) &= 0 \\
&\vdots \\
u_1'(t)y_1^{(n-2)}(t) + u_2'(t)y_2^{(n-2)}(t) + \cdots + u_n'(t)y_n^{(n-2)}(t) &= 0.
\end{aligned}$$

These $n-1$ equations are our constraints on the functions $u_1(t), \ldots, u_n(t)$. Differentiating one final time, we find that

$$\begin{aligned}\frac{d^n\psi}{dt^n} &= u_1(t)y_1^{(n)}(t) + u_2(t)y_2^{(n)}(t) + \cdots + u_n(t)y_n^{(n)}(t) \\ &\quad + u_1'(t)y_1^{(n-1)}(t) + u_2'(t)y_2^{(n-1)}(t) + \cdots + u_n'(t)y_n^{(n-1)}(t).\end{aligned}$$

Now we can plug our formulae for $\psi, d\psi/dt, \ldots, \partial^n\psi/\partial t^n$ into (38):

$$\begin{aligned}&u_1(t)y_1^{(n)}(t) + u_2(t)y_2^{(n)}(t) + \cdots + u_n(t)y_n^{(n)}(t) \\ &+ u_1'(t)y_1^{(n-1)}(t) + u_2'(t)y_2^{(n-1)}(t) + \cdots + u_n'(t)y_n^{(n-1)}(t) \\ &+ p_{n-1}(t)\Big[u_1(t)y_1^{(n-1)}(t) + u_2(t)y_2^{(n-1)}(t) + \cdots + u_n(t)y_n^{(n-1)}(t)\Big] \\ &+ p_{n-2}(t)\Big[u_1(t)y_1^{(n-2)}(t) + u_2(t)y_2^{(n-2)}(t) + \cdots + u_n(t)y_n^{(n-3)}(t)\Big] \\ &\vdots \\ &+ p_1(t)\Big[u_1(t)y_1'(t) + u_2(t)y_2'(t) + \cdots + u_n(t)y_n'(t)\Big] \\ &+ p_0(t)\Big[u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t)\Big] = g(t).\end{aligned}$$

Rearranging then gives

$$\begin{aligned}&u_1'(t)y_1^{(n-1)}(t) + u_2'(t)y_2^{(n-1)}(t) + \cdots + u_n'(t)y_n^{(n-1)}(t) \\ &+ u_1(t)\Big[y_1^{(n)}(t) + p_{n-1}(t)y_1^{(n-1)}(t) + p_{n-2}(t)y_1^{(n-2)}(t) + \cdots + p_1(t)y_1'(t) + p_0(t)y_1(t)\Big] \\ &+ u_2(t)\Big[y_2^{(n)}(t) + p_{n-1}(t)y_2^{(n-1)}(t) + p_{n-2}(t)y_2^{(n-2)}(t) + \cdots + p_1(t)y_2'(t) + p_0(t)y_2(t)\Big] \\ &\vdots \\ &+ u_n(t)\Big[y_n^{(n)}(t) + p_{n-1}(t)y_n^{(n-1)}(t) + p_{n-2}(t)y_n^{(n-2)}(t) + \cdots + p_1(t)y_n'(t) + p_0(t)y_n(t)\Big] = g(t),\end{aligned}$$

and since $y_1(t), y_2(t), \ldots, y_n(t)$ are all solutions of (37), we obtain

$$u_1'(t)y_1^{(n-1)}(t) + u_2'(t)y_2^{(n-1)}(t) + \cdots + u_n'(t)y_n^{(n-1)}(t) = g(t).$$

Combining this equation with the $n-1$ additional constraints we derived above gives us a system of n equations which we can solve algebraically for the functions $u_1'(t), u_2'(t), \ldots, u_n'(t)$. Using Cramer's rule to do so gives us:

$$u_k'(t) = \frac{W_k\left[u_1, u_2, \ldots, u_n\right](t)}{W\left[u_1, u_2, \ldots, u_n\right](t)},$$

where W denotes the Wronskian as defined in section 3.1 and W_k denotes the Wronskian with the kth column of the determinant replaced by the column vector $\langle 0, 0, \ldots, 0, g(t)\rangle$. Therefore, a particular solution to (38) is given by:

$$\psi(t) = \sum_{k=0}^{n}\left[y_k(t)\int \frac{W_k\left[u_1, u_2, \ldots, u_n\right](t)}{W\left[u_1, u_2, \ldots, u_n\right](t)}\,dt\right].$$

3.6. **Method of Judicious Guessing.** We present the results for the method of judicious guessing for higher-order differential equations without proof.

Particular solutions of

$$c_n\frac{d^ny}{dt^n} + \cdots + c_1\frac{dy}{dt} + c_0 y = a_0 + a_1 t + \cdots + a_k t^k$$

occur in the forms

$$\begin{aligned}
&A_0 + A_1 t + \cdots + A_k t^k, && \text{if } c_0 \neq 0,\\
&t\Big[A_0 + A_1 t + \cdots + A_k t^k\Big], && \text{if } c_0 = 0 \text{ but } c_1 \neq 0,\\
&t^2\Big[A_0 + A_1 t + \cdots + A_k t^k\Big], && \text{if } c_0 = c_1 = 0 \text{ but } c_2 \neq 0,\\
&\vdots && \vdots\\
&t^n\Big[A_0 + A_1 t + \cdots + A_k t^k\Big], && \text{if } c_0 = c_1 = \cdots = c_{n-1} = 0 \text{ but } c_n \neq 0.
\end{aligned}$$

If all the coefficients c_i are zero, the differential equation has no solution, so we need not consider this case.

Considering the homogeneous equation

$$c_n \frac{d^n y}{dt^n} + \cdots + c_1 \frac{dy}{dt} + c_0 y = 0, \tag{39}$$

particular solutions of

$$c_n \frac{d^n y}{dt^n} + \cdots + c_1 \frac{dy}{dt} + c_0 y = \Big[a_0 + a_1 t + \cdots + a_k t^k\Big] e^{\gamma t}$$

occur in the forms

$$\begin{aligned}
&\Big[A_0 + A_1 t + \cdots + A_k t^k\Big] e^{\gamma t}, && \text{if } y = e^{\gamma t} \text{ does not solve (39)},\\
&t\Big[A_0 + A_1 t + \cdots + A_k t^k\Big] e^{\gamma t}, && \text{if } y = e^{\gamma t} \text{ solves (39) but } y = te^{\gamma t} \text{ does not},\\
&t^2\Big[A_0 + A_1 t + \cdots + A_k t^k\Big] e^{\gamma t}, && \text{if } y = te^{\gamma t} \text{ solves (39) but } y = t^2 e^{\gamma t} \text{ does not},\\
&\vdots && \vdots\\
&t^n\Big[A_0 + A_1 t + \cdots + A_k t^k\Big] e^{\gamma t}, && \text{if } y = t^{n-1} e^{\gamma t} \text{ solves (39) but } y = t^n e^{\gamma t} \text{ does not}.
\end{aligned}$$

It is impossible for $y = t^n e^{\gamma t}$ to be a solution of (39), but we left this condition in for symmetry.

Particular solutions of

$$c_n \frac{d^n y}{dt^n} + \cdots + c_1 \frac{dy}{dt} + c_0 y = \Big[a_0 + a_1 t + \cdots + a_k t^k\Big] e^{\lambda t} \cos \mu t$$

occur in the forms

$$\begin{aligned}
&\Big[A_0 + A_1 t + \cdots + A_k t^k\Big] e^{\lambda t} \cos(\mu t) + \Big[B_0 + B_1 t + \cdots + B_k t^k\Big] e^{\lambda t} \sin(\mu t),\\
&\qquad \text{if } y = e^{\lambda t} \cos(\mu t) \text{ does not solve (39)},\\
&t\Big[A_0 + A_1 t + \cdots + A_k t^k\Big] e^{\lambda t} \cos(\mu t) + t\Big[B_0 + B_1 t + \cdots + B_k t^k\Big] e^{\lambda t} \sin(\mu t),\\
&\qquad \text{if } y = e^{\lambda t} \cos(\mu t) \text{ solves (39) but } y = te^{\lambda t} \cos(\mu t) \text{ does not},\\
&t^2\Big[A_0 + A_1 t + \cdots + A_k t^k\Big] e^{\lambda t} \cos(\mu t) + t^2\Big[B_0 + B_1 t + \cdots + B_k t^k\Big] e^{\lambda t} \sin(\mu t),\\
&\qquad \text{if } y = te^{\lambda t} \cos(\mu t) \text{ solves (39) but } y = t^2 e^{\lambda t} \cos(\mu t) \text{ does not},\\
&\qquad \vdots\\
&t^n\Big[A_0 + A_1 t + \cdots + A_k t^k\Big] e^{\lambda t} \cos(\mu t) + t^n\Big[B_0 + B_1 t + \cdots + B_k t^k\Big] e^{\lambda t} \sin(\mu t),
\end{aligned}$$

$$\text{if } y = t^{n-1}e^{\lambda t}\cos(\mu t) \text{ solves (39) but } y = t^n e^{\lambda t}\cos(\mu t) \text{ does not.}$$

Again, it is impossible for $y = t^n e^{\gamma t}\cos(\mu t)$ to be a solution of (39), but we left this condition in for symmetry. The result for

$$c_n \frac{d^n y}{dt^n} + \cdots + c_1 \frac{dy}{dt} + c_0 y = \left[a_0 + a_1 t + \cdots + a_k t^k\right] e^{\lambda t}\sin \mu t$$

is analogous.

The discussion in section 2.6.4 also applies to higher-order equations.

3.7. **Power Series Solutions.** Considering the homogeneous nth-order linear differential equation

$$P_n(t)\frac{d^n y}{dt^n} + P_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_1(t)\frac{dy}{dt} + P_0(t)y = 0,$$

we suppose that solutions will be of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n.$$

The analysis is exactly analogous to that of section 2.7.

3.8. **Laplace Transforms.** Suppose that we are given a nonhomogeneous nth-other linear differential equation of the form

$$c_n \frac{d^n y}{dt^n} + c_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + c_1 \frac{dy}{dt} + c_0 y = f(t) \tag{40}$$

and the initial conditions

$$\begin{aligned} y(t_0) &= y_0 \\ y'(t_0) &= y_0' \\ &\vdots \\ y^{(n-1)}(t_0) &= y_0^{(n-1)}. \end{aligned}$$

If the initial conditions are not given at $t = 0$, we may use the transformation discussed in section 2.9.

It can then be shown, using a simple generalization of the method of section 2.9, that a particular solution of (40) is given by

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{\mathcal{L}\{f(t)\} + \sum_{k=0}^{n}\sum_{i=0}^{k-1}\left[c_k s^{k-i-1} y_0^{(i)}\right]}{\sum_{k=0}^{n}\left[c_k s^k\right]} \right\}.$$

3.9. **Transformation to System.** Given the general homogeneous nth-order linear differential equation

$$y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_1(t)y'(t) + p_0(t)y(t) = 0,$$

we can define

$$\mathbf{y}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix}.$$

It then follows that

$$\begin{aligned}
\mathbf{y}'(t) &= \begin{bmatrix} y'(t) \\ y''(t) \\ \vdots \\ y^{(n)}(t) \end{bmatrix} \\
&= \begin{bmatrix} y'(t) \\ \vdots \\ y^{(n-1)}(t) \\ -p_{n-1}(t)y^{(n-1)}(t) - \cdots - p_0(t)y(t) \end{bmatrix} \\
&= y(t) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -p_0(t) \end{bmatrix} + y'(t) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ -p_1(t) \end{bmatrix} + y''(t) \begin{bmatrix} 0 \\ 1 \\ \vdots \\ -p_2(t) \end{bmatrix} + \cdots \\
&\quad + y^{(n-2)}(t) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -p_{n-2}(t) \end{bmatrix} + y^{(n-1)}(t) \begin{bmatrix} 0 \\ \vdots \\ 1 \\ -p_{n-1}(t) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -p_0(t) & -p_1(t) & -p_2(t) & -p_3(t) & \cdots & -p_{n-1}(t) \end{bmatrix} \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \\ \vdots \\ y^{(n-2)}(t) \\ y^{(n-1)}(t) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -p_0(t) & -p_1(t) & -p_2(t) & -p_3(t) & \cdots & -p_{n-1}(t) \end{bmatrix} \mathbf{y}(t).
\end{aligned}$$

That is, we can transform a homogeneous nth-order linear differential equation to a system of n homogeneous first-order linear differential equations.

3.10. **What to Do in General.** The following strategy is advisable:

(1) If the equation is missing a y term, use the strategy of section 3.2.
(2) If the equation is linear and homogeneous and has constant coefficients, use the strategy of section 3.4.
(3) If the equation is linear and homogeneous and has nonconstant coefficients, and you already know one solution, use the strategy of section 3.3 to find another.
(4) If the equation is linear and homogeneous and has nonconstant coefficients that can be expanded as power series, use the strategy of section 3.7.
(5) If the equation is linear and nonhomogeneous, has constant coefficients, and has a right-hand side of the appropriate form, use the strategy of section 3.4 to find the general solution of the corresponding homogeneous equation and then use the strategy of section 3.6.
(6) If the equation is linear and nonhomogeneous and has constant coefficients but does not have a right-hand side admitting of the method of judicious guessing, either (i) use the strategy of

section 3.4 to find the general solution of the corresponding homogeneous equation and then use the strategy of section 3.5, or (ii) use the strategy of section 3.8.